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On a theorem due to Crouzeix and Ferland

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Abstract In this article we introduce the notions of Kuhn-Tucker and Fritz John pseudoconvex nonlinear programming problems with inequality constraints. We derive several properties of these problems. We prove that the problem with quasiconvex data is (second-order) Kuhn-Tucker pseudoconvex if and only if every (second-order) Kuhn-Tucker stationary point is a global minimizer. We obtain respective results for Fritz John pseudoconvex problems. For the first-order case we consider Fréchet differentiable functions and locally Lipschitz ones, for the second-order case Fréchet and twice directionally differentiable functions.

Keywords Nonsmooth analysis · Nonsmooth optimization · Generalized convexity · KT pseudoconvex problems · FJ pseudoconvex problems · Quasiconvex programming

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1 Introduction

In this article we investigate a theorem which appeared in Crouzeix, Ferland [10, Lemma 2.1]. The theorem says that a real Fréchet differentiable quasiconvex function f, defined on an open convex set $S \subseteq \mathbf{R}^n$, is pseudoconvex on S if and only if each stationary point of f is a global minimizer.

Consider the nonlinear programming problem with inequality constraints and a set constraint

$$\text{Minimize} \quad f(x) \quad \text{subject to} \quad x \in X, \; g_i(x) \leq 0, \quad i = 1, 2, \ldots, m \tag{P}$$

where X is an open convex set in the finite dimensional Euclidean space $\mathbf{R}^n$ and the functions $g_i, i = 1, 2, \ldots, m$ are defined on X. We define the notions KT pseudoconvexity and FJ one

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for the problem (P) with locally Lipschitz functions in terms of the Clarke's generalized derivative. We prove that pseudoconvex problems obey some useful properties. We show that the Fréchet differentiable problem with quasiconvex objective function and quasiconvex constraints is KT pseudoconvex if and only if each KT point is a global minimizer. We prove that the locally Lipschitz problem with quasiconvex objective function and strictly quasiconvex constraints is FJ pseudoconvex if and only if every FJ point is a global minimizer. We obtain an extension of the sufficient conditions for a global minimum to problems with quasiconvex objective function and non-quasiconvex constraints. Sufficient conditions for a global minimum are derived in Arrow, Enthoven [1], Mangasarian [26], Bector, Grover [4], Bhatt, Mishra [7], Singh [29], Skarpness, Sposito [30], Bector, Bector [3], Bector, Chandra, Bector [5], Giorgi [16].

In all mentioned articles are given only sufficient criteria for a global minimum, but our conditions are both necessary and sufficient. Necessary and sufficient conditions are obtained in Tanaka, Fukushima, Ibaraki [32, Theorem 3.5] under the assumption that all functions are essentially pseudoconvex, locally Lipschitz and regular in the sense of Clarke, the problem is calm. Another generalization, based on a different idea, is obtained in Ivanov [21].

We define the notions of second-order KT pseudoconvex problem and second-order FJ pseudoconvex one with inequality constraints. We prove some properties of these problems. We prove that the problem (P) with Fréchet and twice directionally differentiable quasiconvex data is second-order KT pseudoconvex if and only if every second-order KT point is a global minimizer. We show that the problem (P) with quasiconvex objective function and strictly quasiconvex constraints is second-order FJ pseudoconvex if and only if each Fritz John point of second-order is a global minimizer. We obtain an extension of the second-order sufficient optimality conditions in Ginchev, Ivanov [13, Theorems 1, 3]. All mentioned results are connected with the theorem of Crouzeix and Ferland. Additionally we give strict variants of all theorems.

The definitions of pseudoconvex problems are based on the standard notion of a pseudoconvex function due to Mangasarian [26, Definition 9.3.1]. Another definition of a pseudoconvex function is available in the literature [19].

Here is the history of the theorem of Crouzeix and Ferland. It appeared in 1982 after some earlier results of Martos [27], Cottle, Ferland [9], Ferland [12, Theorem 12]. In 1983 Komlósi [23, Theorem 2] extended the theorem to radially continuous functions using Dini derivatives. After that other generalizations are obtained by Aussel [2, Theorem 4.1] and Ivanov [20, Theorem 5.1] using subdifferentials. Another characterization of pseudoconvex functions, based on the theorem of Crouzeix and Ferland, is derived independently by Giorgi [15] and Tanaka [31]. Later the last one is generalized by Penot [28, Proposition 13] and Ivanov [20, Theorem 5.2].

In the sequel we denote the scalar product of the vectors $a, b \in \mathbf{R}^n$ by $a\,b$, the closed segment in $\mathbf{R}^n$ with endpoints x and y by $[x, y]$ and the respective open segment by (x, y), the closed non-negative orthant in $\mathbf{R}^n$ by $\mathbf{R}^n_+$. We use the notation := for "equal by definition". We denote by g the vector function with components g_i, $i = 1, 2, \ldots, m$. The notation $f \in C^1(X)$, $g \in C^1(X)$ implies that f and g are continuously differentiable on X.

The article is organized as follows: In Sect. 2 we define the notions of KT and FJ pseudoconvex problems. We obtain the first-order extensions of the theorem of Crouzeix and Ferland using these definitions. In Sect. 3 we define the notions of second-order KT and FJ pseudoconvex problems. Then we apply these definitions in the second-order extensions of the theorem.

2 First-order extensions

2.1 KT pseudoconvex problems

We begin this section with some well-known definitions:

Definition 2.1 Let $X \subseteq \mathbf{R}^n$ be a convex set. A function $f : X \to \mathbf{R}$ is called:

(i) quasiconvex on X if
$f(x + t(y - x)) \leq \max(f(x), f(y)), \quad \forall x \in X, \forall y \in X, \forall t \in [0, 1];$

(ii) semistrictly quasiconvex on X if
$x, y \in X$, $f(y) < f(x)$ imply $f(x + t(y - x)) < f(x), \quad \forall t \in (0, 1);$

(iii) strictly quasiconvex on X if
$f(x + t(y - x)) < \max(f(x), f(y)), \quad \forall x \in X, \forall y \in X, \forall t \in (0, 1).$

Every strictly quasiconvex function is semistrictly quasiconvex and every lower semicontinuous semistrictly quasiconvex function is quasiconvex.

The following definition about quasiconvexity for differentiable functions is used in the sufficient conditions for a global minimum:

Definition 2.2 Let X be an open set and f differentiable at $x \in X$. Then f is said to be quasiconvex at x with respect to X if

$$y \in X, \ f(y) \leq f(x) \quad \text{imply} \quad \nabla f(x)(y - x) \leq 0. \tag{2.1}$$

A differentiable function f is called quasiconvex on X if this implication holds for all $x \in X$.

If X is open and convex, and f differentiable on X, then both definitions about quasiconvexity are equivalent (see Arrow, Enthoven [1]).

Definition 2.3 Let $X \subseteq \mathbf{R}^n$ be an open set and f a locally Lipschitz function, defined on X. The Clarke's generalized derivative [8] of f at the point $x \in X$ in the direction $d \in \mathbf{R}^n$ is defined as follows:

$$f^0(x; d) := \limsup_{(t,x') \to (+0,x)} t^{-1}(f(x' + td) - f(x'))$$

and the Clarke's generalized gradient of f at x by

$$\partial f(x) := \{\xi \in \mathbf{R}^n \mid \xi d \leq f^0(x; d), \ \forall d \in \mathbf{R}^n\}.$$

The Clarke's generalized gradient is a nonempty compact set and therefore it is linked to the directional derivative by the following equality:

$$f^0(x; d) = \max\{\xi d \mid \xi \in \partial f(x)\} \quad \text{for all} \quad d \in \mathbf{R}^n. \tag{2.2}$$

The Clarke's derivative satisfies several useful properties. The following one is among them:

$$f^0(x; -d) = (-f)^0(x; d) \quad \text{for all} \quad x \in X, \ d \in \mathbf{R}^n. \tag{2.3}$$

Definition 2.4 Let $X \subseteq \mathbf{R}^n$ be an open set. The function f, defined on X, is said to be (strictly) pseudoconvex with respect to the Clarke's generalized derivative (for short, (strictly) pseudoconvex) at $x \in X$ if

$$y \in X \text{ and } f(y) < f(x)\ (f(y) \leq f(x), y \neq x) \quad \text{imply} \quad f^0(x; y-x) < 0.$$

f is said to be (strictly) pseudoconvex on X if this implication holds for all $x \in X$. It follows from relation (2.2) that this definition is equivalent to the following implication:

$$f(y) < f(x)\ (f(y) \leq f(x), y \neq x) \quad \Rightarrow \quad \xi(y-x) < 0,\ \forall \xi \in \partial f(x).$$

The function f is said to be (strictly) pseudoconcave if $(-f)$ is (strictly) pseudoconvex.

The next claim is a generalization of a result due to Karamardian [22] concerning Fréchet differentiable functions.

Proposition 2.1 *Let the function f be locally Lipschitz pseudoconvex with respect to the Clarke's generalized derivative on the open convex set $X \subseteq \mathbf{R}^n$. Then f is semistrictly quasiconvex on X.*

Proof Suppose the contrary that f is not semistrictly quasiconvex. Therefore there exist $x,\ y \in X$ and $z = x + \lambda(y - x)$, $\lambda \in (0, 1)$ such that $f(y) < f(x) \leq f(z)$. It follows from pseudoconvexity that $f^0(z; y - z) < 0$. Using the relation $z - x = \frac{\lambda}{1-\lambda}(y - z)$ and (2.3) we obtain that

$$(-f)^0(z; x-z) = f^0(z; z-x) = \frac{\lambda}{1-\lambda} f^0(z; y-z) < 0 \tag{2.4}$$

Since the function $(-f)$ is pseudoconcave, we conclude from here that $(-f)(x) \leq (-f)(z)$. Hence $f(y) < f(x) = f(z)$. It follows from (2.4) that there exists $\tau > 0$ with

$$(-f)(z + t(x-z)) < (-f)(z) \quad \text{for all} \quad t \in (0, \tau).$$

Take arbitrary $t \in (0, \tau)$. Denote $u = z + t(x - z)$. Since the Clarke's generalized gradient is a nonempty set, then there exists $\xi \in \partial f(u)$. Thanks to $f(y) < f(x) < f(u)$ we have that

$$\xi(x-u) \leq f^0(u; x-u) < 0 \quad \text{and} \quad \xi(y-u) \leq f^0(u; y-u) < 0$$

which is impossible. □

A similar claim holds for strictly pseudoconvex functions.

Proposition 2.2 *Each locally Lipschitz strictly pseudoconvex function, defined on an open convex set X, is strictly quasiconvex.*

Proof Assume the contrary that there exist $x,\ y \in X$ and $z \in (x, y)$ with

$$f(y) \leq f(x) \leq f(z).$$

Since $\partial f(z) \neq \emptyset$ we have that there exists $\xi \in \partial f(z)$. By the strict pseudoconvexity of f we obtain

$$\xi(x-z) \leq f^0(z; x-z) < 0, \quad \xi(y-z) \leq f^0(z; y-z) < 0.$$

These inequalities contradict each other. □

The next claim is a particular case of Theorem 2.1 in Aussel [2].

Lemma 2.1 *Let the function f be locally Lipschitz on the open convex set $X \subseteq \mathbf{R}^n$. Then f is quasiconvex on X if and only if for all $x, y \in X$ the following implication holds:*

$$f^0(x; y-x) > 0 \quad \Rightarrow \quad f(z) \le f(y), \ \forall z \in [x, y].$$

It follows from here that each quasiconvex function satisfies the implication

$$x, y \in X, \ f(y) < f(x) \quad \Rightarrow \quad f^0(x; y-x) \le 0.$$

When the function is Fréchet differentiable, then implication (2.1) is fulfilled.

The following simple example shows that implication (2.1) fails for locally Lipschitz quasiconvex functions when the Clarke's generalized gradient is used.

Example 2.1 Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if} \quad x \ge 0, \\ x, & \text{if} \quad x < 0. \end{cases}$$

It is locally Lipschitz and quasiconvex. We have $f^0(0; d) = d$ if $d \ge 0$, and $f^0(0; d) = 0$ if $d < 0$. Therefore $\partial f(0) = [0, 1]$ and $f(y) = f(0) = 0$ if $y > 0$, but $\xi y > 0$ if $y > 0$ and $\xi > 0$.

On the other hand the following result due to Daniilidis, Hadjisavvas [11, Theorem 3.1] shows that property (2.1) holds when the function is locally Lipschitz semistrictly quasiconvex and the Clarke's generalized gradient is applied.

Lemma 2.2 *A locally Lipschitz function f, defined on an open convex set X in $\mathbf{R}^n$, is semistrictly quasiconvex on X if and only if for all $x, y \in X$ the following implication holds:*

$$f^0(x; y-x) > 0 \quad \Rightarrow \quad f(z) < f(y), \ \forall z \in [x, y).$$

It follows from here that every locally Lipschitz semistrictly quasiconvex function satisfies the following implication:

$$x, y \in X, \ f(y) \le f(x) \quad \Rightarrow \quad f^0(x; y-x) \le 0. \tag{2.5}$$

It is proved in Komlósi [24, Theorem 4.4] that for a function which is not a constant on any line segment $[x, y]$ of its domain (radially non-constant function) implication (2.5) is equivalent to quasiconvexity. On the other hand a function is strictly quasiconvex if and only if it is both quasiconvex and radially non-constant.

We denote the set of all feasible points of (P) by

$$S := \{x \in X \mid g_i(x) \le 0, \quad i = 1, 2, \ldots, m\}.$$

For every $x \in S$ let $I(x)$ be the set of the active constraints

$$I(x) := \{i \in \{1, 2, \ldots, m\} \mid g_i(x) = 0\}.$$

We call the problem (P) locally Lipschitz if f and $g_i, i = 1, 2, \ldots, m$ are locally Lipschitz. The following result is well-known (see Clarke [8, Theorem 6.1.1]).

Lemma 2.3 *(*Necessary condition for a local minimum*) Let X be an open set in the space $\mathbf{R}^n$. Suppose that x is a local minimizer of* (P)*, the functions f, g_i, $i \in I(x)$ are locally*

Lipschitz and the functions g_i, $i \notin I(x)$ *are continuous. Then there exist Lagrange multipliers* $\lambda \geq 0$, $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbf{R}^m_+$, *with* $(\lambda, \mu) \neq (0, 0)$ *and* $a \in \partial f(x)$, $b_i \in \partial g_i(x)$, $i \in I(x)$ *such that*

$$\lambda a + \sum_{i \in I(x)} \mu_i b_i = 0, \;\; \mu_i g_i(x) = 0, \quad i = 1, 2, \ldots, m. \tag{2.6}$$

If additionally we assume that a constraint qualification holds (*see Hiriart-Urruty* [18]), *then we can suppose that* $\lambda > 0$.

Definition 2.5 If there exist $(\lambda, \mu) \in ([0, +\infty) \times \mathbf{R}^m_+) \setminus \{(0, 0)\}$, $a \in \partial f(x)$, and $b_i \in \partial g_i(x)$, $i \in I(x)$ such that equations (2.6) hold, then the point $x \in S$ is called a Fritz John stationary point (for short, FJ point). If additionally $\lambda = 1$, then x is called a Kuhn-Tucker stationary point (for short, KT point).

The following claim is a simple consequence of Definition 2.5:

Lemma 2.4 *Let the problem* (P) *be locally Lipschitz and* $x \in X$.

(a) *If* $0 \in \partial f(x)$, *then* x *is a KT point.*
(b) *If* $0 \in \partial g_i(x)$ *for some* $i \in I(x)$, *then* x *is a FJ point.*

The following assertions play crucial role in several proofs below.

Lemma 2.5 *Let the function* f *be locally Lipschitz on the open convex set* $X \subseteq \mathbf{R}^n$.

(a) *If* f *is quasiconvex on* X, $x, y \in X$, $f(y) < f(x)$ *and* $f^0(x; y - x) = 0$, *then* $0 \in \partial f(x)$. *In particular, if* f *is Fréchet differentiable quasiconvex on* X *and* $x, y \in X$, $f(y) < f(x)$, $\nabla f(x)(y - x) = 0$, *then* $\nabla f(x) = 0$.
(b) *If* f *is strictly quasiconvex on* X, $x, y \in X$, $x \neq y$, $f(y) \leq f(x)$ *and* $f^0(x; y - x) = 0$, *then* $0 \in \partial f(x)$. *In particular, if* f *is Fréchet differentiable strictly quasiconvex on* X *and* $x, y \in X$, $x \neq y$, $f(y) \leq f(x)$, $\nabla f(x)(y - x) = 0$, *then* $\nabla f(x) = 0$.

Proof We prove Claim (a). It follows from relation (2.2) that $f^0(x; y - x) = \max\{\xi(y - x) \mid \xi \in \partial f(x)\}$. Therefore there exists $a^* \in \partial f(x)$ such that $a^*(y - x) = f^0(x; y - x) = 0$. By the continuity of f we obtain from the inequality $f(y) < f(x)$ that there exists $\delta > 0$ with $f(y + \delta a^*) < f(x)$. According to Lemma 2.1 we conclude that $a^*(y + \delta a^* - x) \leq 0$. Then $a^*(y - x) = 0$ implies that $a^* = 0$. Hence $0 \in \partial f(x)$.

Claim (b). It follows from relation (2.2) that there exists $a^* \in \partial f(x)$ with $a^*(y - x) = 0$. Choose arbitrary $z \in (x, y)$. By strict quasiconvexity $f(z) < f(x)$. There exists $\delta > 0$ such that $f(z + \delta a^*) < f(x)$. Using the arguments of Claim (a) we obtain that $a^* = 0$. □

We consider the following optimality conditions for the problem with inequality constraints given in [26, Theorem 10.1.1].

Proposition 2.3 *Let* X *be an open set in* $\mathbf{R}^n$. *Suppose that* f *is Fréchet differentiable and pseudoconvex at* x, g_i, $i \in I(x)$ *are Fréchet differentiable and quasiconvex at* x. *If* $x \in S$ *is a KT point of the problem* (P), *then it is a global minimizer.*

It is easy to see that we can relax the hypothesis of this proposition. For x to be a global minimizer it is enough to suppose that the following implications hold together:

$$y \in S, \;\; f(y) < f(x) \quad \Rightarrow \quad \nabla f(x)(y - x) < 0$$

and

$$y \in S,\ i \in I(x) \quad \Rightarrow \quad \nabla g_i(x)(y-x) \le 0$$

which is weaker than the assumption f is pseudoconvex at x and $g_i, i \in I(x)$ are quasiconvex at x separately.

We introduce the following definition:

Definition 2.6 We call the problem (P) (strictly) KT pseudoconvex at the point $x \in S$ with respect to the Fréchet derivative if the following implication holds:

$$\left.\begin{array}{l} y \in S, \\ f(y) < f(x) \end{array}\right] \Rightarrow \left[\begin{array}{l} \nabla f(x)(y-x) < 0, \\ \nabla g_i(x)(y-x) \le 0 \ \text{ for all } \ i \in I(x) \end{array}\right.$$

$$\left(\left.\begin{array}{l} y \in S,\ y \ne x, \\ f(y) \le f(x) \end{array}\right] \Rightarrow \left[\begin{array}{l} \nabla f(x)(y-x) < 0, \\ \nabla g_i(x)(y-x) \le 0 \ \text{ for all } \ i \in I(x). \end{array}\right.\right)$$

We call (P) (strictly) KT pseudoconvex if it is (strictly) KT pseudoconvex at each $x \in S$.

It is obvious that if (P) is (strictly) KT pseudoconvex, then the objective function is (strictly) pseudoconvex on S. On the other hand, even in the classical Fréchet differentiable case, the condition (P) is KT pseudoconvex does not imply that the constraints are quasiconvex.

Example 2.2 Consider the problem (P) with $f, g : \mathbf{R} \to \mathbf{R}$ where

$$f = x^2,\ g(x) = x^4 - 5x^2 + 4,\ X = (-\infty, +\infty).$$

It is seen that (P) is KT pseudoconvex, but g is not quasiconvex. Indeed, $g(x) = 0$ if $x = \pm 1, \pm 2$. If $x = \pm 1$, then there is no a feasible point y such that $f(y) < f(x)$.

Theorem 2.1 *Let $X \subseteq \mathbf{R}^n$ be an open and convex set. Suppose that f is Fréchet differentiable and (strictly) quasiconvex on X, g_i, $i = 1, 2, \ldots, m$ are Fréchet differentiable and quasiconvex on X. Then the problem* (P) *is (strictly) pseudoconvex if and only if each KT point is a (strict) global minimizer.*

Proof We prove the non-strict case. The proof of the strict one is similar.

Let every KT point be a global minimizer. We prove that (P) is KT pseudoconvex. Suppose that $x, y \in S$ and $f(y) < f(x)$. We have to prove that $\nabla f(x)(y-x) < 0$. Assume the contrary that $\nabla f(x)(y-x) \ge 0$. It follows from the quasiconvexity of f that $\nabla f(x)(y-x) = 0$. By Lemma 2.5 $\nabla f(x) = 0$. We conclude from Lemma 2.4 that x is a KT point. By the hypothesis of the theorem x is a global minimizer which contradicts the assumption $f(y) < f(x)$.

The claim $\nabla g_i(x)(y-x) \le 0$ follows directly from the assumption $g_i(y) \le 0 = g_i(x)$, $i \in I(x)$ and the quasiconvexity of g_i.

At last, we prove the only if part. Suppose that (P) is KT pseudoconvex and x is an arbitrary KT point. Assume that x is not a global minimizer. Hence there is $y \in S$ with $f(y) < f(x)$. Since x is a KT point there exist $\lambda > 0$ and $\mu_i \ge 0$, $i \in I(x)$ which satisfy the following equation:

$$\lambda \nabla f(x) + \sum_{i \in I(x)} \mu_i \nabla g_i(x) = 0. \tag{2.7}$$

Then we multiply the inequality $\nabla f(x)(y-x) < 0$ by λ and $\nabla g_i(x)(y-x) \le 0$ by μ_i and add all obtained inequalities. Thus we conclude from (2.7) that $0 < 0$ which is impossible. Therefore x is a global minimizer. □

Definition 2.7 We call the locally Lipschitz problem (P) (strictly) KT pseudoconvex at the point $x \in S$ with respect to the Clarke's generalized derivative if the following implication holds

$$\left.\begin{matrix} y \in S, \\ f(y) < f(x) \end{matrix}\right] \Rightarrow \left[\begin{matrix} f^0(x; y-x) < 0, \\ g_i^0(x; y-x) \leq 0 \ \text{ for all } \ i \in I(x). \end{matrix}\right.$$

$$\left(\left.\begin{matrix} y \in S, \ y \neq x, \\ f(y) \leq f(x) \end{matrix}\right] \Rightarrow \left[\begin{matrix} f^0(x; y-x) < 0, \\ g_i^0(x; y-x) \leq 0 \ \text{ for all } \ i \in I(x). \end{matrix}\right. \right)$$

We call (P) (strictly) KT pseudoconvex if it is (strictly) KT pseudoconvex at each $x \in S$.

Theorem 2.2 *Let X be an open set in $\mathbf{R}^n$. Suppose that the problem* (P) *is locally Lipschitz and (strictly) KT pseudoconvex. Then*

(a) *each KT point is a (strict) global minimizer;*
(b) *every local minimizer of* (P) *is a global one, provided that a constraint qualification holds* [18].

Proof Claim (a). We consider the non-strict case. Suppose that x is a KT point, but it is not a global minimizer. Therefore there exists $y \in S$ with $f(y) < f(x)$. Since x is a KT point, we obtain that there are $(\lambda, \mu) \in (0, +\infty) \times \mathbf{R}_+^m$ and $a \in \partial f(x)$, $b_i \in \partial g_i(x)$, $i \in I(x)$ such that equations (2.6) hold. On the other hand, by the KT pseudoconvexity of (P) we have $a(y-x) < 0$ and $b_i(y-x) \leq 0$, $i \in I(x)$. Then

$$\left(\lambda a + \sum_{i \in I(x)} \mu_i b_i \right) (y - x) < 0$$

which contradicts the equations (2.6).

Claim (b) follows from Lemma 2.3 and Claim (a). □

Claim (a) generalizes the sufficient optimality conditions in [1,3,4,7,16,26,29] when the problem does not contain equality constraints.

Theorem 2.3 *Let X be an open convex set and the problem* (P) *be locally Lipschitz. Suppose additionally that f is (strictly) quasiconvex and g_i, $i = 1, 2, \ldots, m$ are semistrictly quasiconvex. Then* (P) *is (strictly) KT pseudoconvex if and only if each KT point is a (strict) global minimizer.*

Proof We consider the non-strict case. Assume that each KT point of (P) is a global minimizer. We prove that the program (P) is KT pseudoconvex. Let $x, y \in S$ be such that $f(y) < f(x)$. Following this aim we prove that $f^0(x; y-x) < 0$. Assume the contrary that $f^0(x; y-x) \geq 0$. According to Lemma 2.1 we obtain that $f^0(x; y-x) = 0$. Then we obtain from Lemma 2.5 that $0 \in \partial f(x)$. It follows from Lemma 2.4 that x is a KT point. By the hypothesis x is a global minimizer which contradicts the assumption $f(y) < f(x)$.

The claim $g_i^0(x; y-x) \leq 0$ for $i \in I(x)$ follows from Lemma 2.2.

The only if part is a Theorem 2.2, Claim (a). □

The following result is a strict variant of the Theorem of Crouzeix and Ferland.

Corollary 1 *Let the function f be locally Lipschitz and strictly quasiconvex on the open convex set X. Then f is strictly pseudoconvex if and only if each point x such that $0 \in \partial f(x)$ is a strict global minimizer.*

Proof The proof follows from the strict case when the problem has no constraints. Let $0 \in \partial f(x)$ imply that x is a strict global minimizer. Suppose that $y \in X$, $y \neq x$, $f(y) \leq f(x)$ and $f^0(x; y-x) \geq 0$. It follows from strict quasiconvexity that $f^0(x; y-x) = 0$. By Lemma 2.5 we have $0 \in \partial f(x)$ which is impossible. □

The following example shows that we cannot replace in Theorem 2.3 the condition g_i, $i \in I(x)$ are semistrictly quasiconvex by g_i, $i \in I(x)$ are quasiconvex.

Example 2.3 Consider the problem (P) where $f, g : \mathbf{R} \to \mathbf{R}$ and $X = (-1, 1)$. We take $f(x) = -x$ and $g(x)$ is the function from Example 2.1. It is obvious that f is pseudoconvex, g is quasiconvex, but not strictly quasiconvex, and $x = 0$ is a KT point, but it is not a global minimizer.

Theorem 2.4 *Let the set X be open and convex. Suppose that* (P) *is locally Lipschitz. If the objective function f is* (*strictly*) *quasiconvex and each KT point of* (P) *is a* (*strict*) *global minimizer, then f is* (*strictly*) *pseudoconvex on S.*

Proof The proof repeats some of the arguments of Theorem 2.3. □

2.2 FJ pseudoconvex problems

We introduce the following definition:

Definition 2.8 We call the problem (P) (strictly) FJ pseudoconvex at the point $x \in S$ with respect to the Clarke's generalized derivative if the following implication holds:

$$\left.\begin{matrix} y \in S, \\ f(y) < f(x) \end{matrix}\right] \quad \Rightarrow \quad \left[\begin{matrix} f^0(x; y-x) < 0, \\ g_i^0(x; y-x) < 0 \ \text{ for all } \ i \in I(x). \end{matrix}\right.$$

$$\left(\left.\begin{matrix} y \in S, \ y \neq x, \\ f(y) \leq f(x) \end{matrix}\right] \quad \Rightarrow \quad \left[\begin{matrix} f^0(x; y-x) < 0, \\ g_i^0(x; y-x) < 0 \ \text{ for all } \ i \in I(x). \end{matrix}\right. \right)$$

We call (P) (strictly) FJ pseudoconvex if it is (strictly) FJ pseudoconvex at each $x \in S$.

The notion (strict) FJ pseudoconvexity implies (strict) KT pseudoconvexity.

FJ Pseudoconvex problems possess some useful properties.

Theorem 2.5 *Let X be an open convex set. Suppose that the problem* (P) *is locally Lipschitz and strictly FJ pseudoconvex. Then the feasible set S is convex.*

Proof Assume the contrary that there exist $x, y \in S$ and $z \in (x, y)$ such that $z \notin S$. Using that all constraint functions are continuous we obtain that the set $[x, y] \cap S$ is closed. Therefore there is an open interval $(u, v) \subset [x, y]$ containing z such that $(u, v) \cap S = \emptyset$. We suppose that (u, v) is the union of all open subintervals of $[x, y]$ which include z and they do not intersect S. Without loss of generality we can suppose that $f(v) \leq f(u)$.

We prove that $u \in S$. By the maximality of (u, v) every neighborhood of u contains a point from $[x, y] \cap S$. Therefore there exists an infinite sequence $\{\alpha_k\}$ such that $\alpha_k \in [x, y] \cap S$ and $\alpha_k \to u$. We conclude from here that $u \in S$. Using similar arguments we prove that $v \in S$.

We prove that there exists an infinite sequence $\{u_k\}$ and an index $j \in \{1, 2, \ldots, m\}$ such that $g_j(u) = 0$, $u_k \in (u, v)$, $u_k \to u$. Indeed, choose arbitrary infinite sequence $u_k \in (u, v)$, $u_k \to u$. For every integer k, by $u_k \notin S$, there exists an index i such that $g_i(u_k) \leq 0$ is not satisfied. Therefore there exists an index j and an infinite subsequence of $\{u_k\}$ such that

$g_j(u_k) > 0$ for every term of this subsequence. Without loss of generality we denote this subsequence by $\{u_k\}$ again. It follows from $u \in S$, $u_k \to u$, $g_j(u_k) > 0$ that $g_j(u) = 0$.

By the strict FJ pseudoconvexity of (P) we have that $g_j^0(u; v - u) < 0$. Hence

$$\limsup_{t \to +0} t^{-1}(g_j(u + t(v - u)) - g_j(u)) < 0$$

and there exist $\delta > 0$ with $g_j(u + t(v - u)) < g_j(u)$ for all $t \in (0, \delta)$ which contradicts the choice of $\{u_k\}$. □

Example 2.2 shows that the assumption (P) is strictly FJ pseudoconvex is essential in Theorem 2.5. Really, (P) is FJ pseudoconvex, but not strictly FJ pseudoconvex and S is not convex.

Theorem 2.6 *Let X be an open set in $\mathbf{R}^n$. Suppose that the problem* (P) *is locally Lipschitz and (strictly) FJ pseudoconvex. Then*

(a) *each FJ point is a (strict) global minimizer;*
(b) *every local minimizer of* (P) *is a global one.*

Proof The proof of Claim (a) is similar to the proof of Theorem 2.2.

Claim (b) follows from Lemma 2.3 and Claim (a). □

Claim (a) extends the respective results in [5,30] when the problem does not contain equality constraints.

Theorem 2.7 *Let X be an open convex set and the problem* (P) *be locally Lipschitz. Suppose additionally that f is (strictly) quasiconvex and g_i, $i = 1, 2, \ldots, m$ are strictly quasiconvex. Then* (P) *is (strictly) FJ pseudoconvex if and only if each FJ point is a (strict) global minimizer.*

Proof Consider the non-strict case. Assume that every FJ point of (P) is a global minimizer. We prove that (P) is FJ pseudoconvex. Let $x, y \in S$ be such that $f(y) < f(x)$.

First, we prove that $f^0(x; y - x) < 0$. Assume the contrary. It follows from Lemma 2.1 that $f^0(x; y - x) = 0$. Then by Lemma 2.5 we obtain that $0 \in \partial f(x)$. According to Lemma 2.4 x is a FJ point. By the hypothesis x is a global minimizer which contradicts the assumption $f(y) < f(x)$.

Second, we prove that $g_i^0(x; y - x) < 0$ for $i \in I(x)$. Suppose the contrary that $g_i^0(x; y - x) \geq 0$. By Lemma 2.2 we obtain that $g_i^0(x; y - x) = 0$. Thanks to Lemma 2.5 we have $0 \in \partial g_i(x)$. Using Lemma 2.4 we conclude that x is FJ point. Therefore it is a global minimizer which contradicts the assumption $y \in S$ and $f(y) < f(x)$.

The only if part of the non-strict case follows from Theorem 2.6, Claim (a). □

3 Second-order extensions

3.1 Second-order KT pseudoconvex problems

Recall the following preliminary definitions:

Definition 3.1 Let $f : \mathbf{R}^n \to \mathbf{R}$ be a Fréchet differentiable function. The limit

$$f''(x; d) := \lim_{t \to +0} 2t^{-2} (f(x + td) - f(x) - t\nabla f(x)d)$$

is called the second-order directional derivative of f at the point $x \in \mathbf{R}^n$ in the direction $d \in \mathbf{R}^n$.

Definition 3.2 Let the functions f and g_i, $i = 1, 2, \ldots, m$ be Fréchet differentiable. A direction d is called critical at the point $x \in S$ [6] if $\nabla f(x)d \leq 0$ and $\nabla g_i(x)d \leq 0$ for all $i \in I(x)$.

If a direction d is not critical at $x \in S$, then $\nabla f(x)d > 0$ or there exists $i \in I(x)$ such that $\nabla g_i(x)d > 0$. Therefore there is $\tau > 0$ with $f(x + td) > f(x)$ for all $t \in (0, \tau)$ or $g_i(x + td) > 0$ for some $i \in I(x)$ and for all $t \in (0, \tau)$. If $\bar{x}$ is a local minimizer and d is a critical direction, then it is possible that there exists a sequence $\{t_k\}$, $t_k > 0$, $t_k \to 0$ with $x + t_k d \in S$ and $f(x + t_k d) = f(x)$.

The following definition was recently introduced by Ginchev, Ivanov [13].

Definition 3.3 Consider a real function f with an open domain X, which is Fréchet differentiable at $x \in X$ and second-order directionally differentiable at $x \in X$ in every direction $y - x$ such that $y \in X$, $f(y) < f(x)$, $\nabla f(x)(y - x) = 0$. Then f is called second-order (strictly) pseudoconvex at $x \in X$ if for all $y \in X$ the following implications hold:

$$f(y) < f(x) \Rightarrow \nabla f(x)(y - x) \leq 0;$$
$$f(y) < f(x),\ \nabla f(x)(y - x) = 0 \Rightarrow f''(x; y - x) < 0.$$
$$(f(y) \leq f(x),\ y \neq x,\ \nabla f(x)(y - x) = 0 \Rightarrow f''(x; y - x) < 0.)$$

The function f is called second-order (strictly) pseudoconvex on X if both implications hold for all $x \in X$.

It follows from this definition that every differentiable pseudoconvex function is second-order pseudoconvex. The converse does not hold.

The following result is a particular case of Theorem 4 in Ginchev, Ivanov [13].

Proposition 3.1 *Let f be a Fréchet differentiable and second-order directionally differentiable second-order pseudoconvex function, defined on an open convex set $X \subseteq \mathbf{R}^n$. Then f is quasiconvex, and moreover, f is semistrictly quasiconvex.*

On the other hand a similar result holds for second-order strictly pseudoconvex functions.

Proposition 3.2 *Let f be a Fréchet differentiable and second-order directionally differentiable second-order strictly pseudoconvex function, defined on the open convex set $X \subseteq \mathbf{R}^n$. Then f is strictly quasiconvex.*

Proof Assume the contrary that there exist $x, y \in X$ and $z \in (x, y)$ such that

$$f(y) \leq f(x) \leq f(z).$$

According to the second-order strict pseudoconvexity of f we conclude that

$$\nabla f(z)(x - y) = \nabla f(z)(y - x) = 0.$$

Applying the second-order strict pseudoconvexity again we obtain that $f''(z; x - y) < 0$, $f''(z; y - x) < 0$. Therefore there exists $\delta > 0$ such that $f(z + t(x - y)) < f(z)$ and $f(z + t(y - x)) < f(z)$ for all $t \in (0, \delta)$ which contradict the quasiconvexity of f. □

The following result is due to Ginchev and Ivanov [14, Theorem 6].

Lemma 3.1 (Second order necessary conditions for optimality) *Let X be an open set in the space $\mathbf{R}^n$, the functions f, g_i $(i = 1, 2, \dots, m)$ be defined on X. Suppose that x is a local minimizer of the problem* (P) *, the functions g_i, $i \notin I(x)$ are continuous at x, the functions f, g_i, $i \in I(x)$ are continuously differentiable, and for every direction $d \in \mathbf{R}^n$ such that $\nabla f(x)d = 0$ and $\nabla g_i(x)d = 0$, $i \in I(x)$ there exist the second-order directional derivatives $f''(x; d)$ and $g_i''(x; d)$, $i \in I(x)$. Then corresponding to any critical direction d there exist non-negative multipliers $\lambda, \mu_1, \dots, \mu_m$, with $(\lambda, \mu) \neq (0, 0)$ such that*

$$\mu_i g_i(x) = 0,\ i = 1, 2, \dots, m, \quad \nabla L(x) = 0, \tag{3.1}$$

$$\mu_i \nabla g_i(x)d = 0, i \in I(x), \quad \lambda \nabla f(x)d = 0,$$

$$L''(x, d) = \lambda f''(x, d) + \sum_{i \in I(x)} \mu_i g_i''(x, d) \geq 0. \tag{3.2}$$

Here $L = \lambda f + \sum_{i=1}^{n} \mu_i g_i$ is the Lagrange function. Assume further that the Guinard constraint qualification holds [17]. *Then we could suppose that $\lambda = 1$.*

Definition 3.4 Let $x \in S$ where S is the feasible set of the problem (P). Suppose that the functions f, g_i $(i = 1, 2, \dots, m)$ are defined on X, Fréchet differentiable, and second-order directionally differentiable at any $x \in S$ in every critical direction $d \in \mathbf{R}^n$. If for every critical direction d there exists

$$(\lambda, \mu) \in ([0, +\infty) \times \mathbf{R}^m_+) \setminus \{(0, 0)\}$$

such that the equations (3.1) and (3.2) are satisfied, then x is called a second-order Fritz John stationary point (for short, FJ point). If additionally $\lambda = 1$, then x is called a second-order Kuhn-Tucker stationary point (for short, KT point).

The following claim is a simple consequence of Definition 3.4:

Lemma 3.2 *Let x be a point from the feasible set S. Suppose that the functions f and g_i, $i \in I(x)$ are Fréchet differentiable and second-order directionally differentiable at x in every direction d.*

(a) *If $\nabla f(x) = 0$ and $f''(x; d) \geq 0$ for all critical directions $d \in \mathbf{R}^n$, then x is a second-order KT point.*
(b) *If $\nabla g_i(x) = 0$ for some $i \in I(x)$ and $g_i''(x; d) \geq 0$ for all critical directions $d \in \mathbf{R}^n$, then x is a second-order FJ point.*

We introduce the following definition:

Definition 3.5 We call the problem (P) second-order (strictly) KT pseudoconvex at the point $x \in S$ if the following implication holds:

$$\left.\begin{array}{l} y \in S, \\ f(y) < f(x) \end{array}\right] \Rightarrow \left[\begin{array}{l} \nabla f(x)(y - x) \leq 0, \\ \nabla f(x)(y - x) = 0 \ \text{ implies } \ f''(x; y - x) < 0, \\ \nabla g_i(x)(y - x) \leq 0 \ \text{ for all } \ i \in I(x), \\ \nabla g_i(x)(y - x) = 0,\ i \in I(x) \ \text{ imply } \ g_i''(x; y - x) \leq 0. \end{array}\right.$$

$$\left(\left.\begin{array}{l} y \in S,\ y \neq x, \\ f(y) \leq f(x) \end{array}\right] \Rightarrow \left[\begin{array}{l} \nabla f(x)(y - x) \leq 0, \\ \nabla f(x)(y - x) = 0 \ \text{ implies } \ f''(x; y - x) < 0, \\ \nabla g_i(x)(y - x) \leq 0 \ \text{ for all } \ i \in I(x), \\ \nabla g_i(x)(y - x) = 0,\ i \in I(x) \ \text{ imply } \ g_i''(x; y - x) \leq 0. \end{array}\right. \right)$$

provided that all necessary derivatives exist. We call (P) second-order (strictly) KT pseudoconvex if it is second-order (strictly) KT pseudoconvex at each $x \in S$.

If (P) is second-order (strictly) KT pseudoconvex, then the objective function f is second-order (strictly) pseudoconvex on S. On the other hand Example 2.2 shows that the property (P) is second-order pseudoconvex does not imply that the constraints are quasiconvex.

Like KT pseudoconvex problems, second-order KT pseudoconvex ones possess some useful properties.

Theorem 3.1 *Let X be an open set. Suppose that the functions f, $g_i (i = 1, 2, \ldots, m)$ are Fréchet differentiable, and second-order directionally differentiable at any $x \in S$ in every critical direction $d \in \mathbf{R}^n$. Assume that* (P) *is second-order (strictly) KT pseudoconvex. Then*

(a) *each KT point of second-order is a (strict) global minimizer;*
(b) *every local minimizer of* (P) *is a global one, provided that $f, g \in C^1(X)$ and a constraint qualification holds* [17].

Proof Claim (a). Let (P) be second-order KT pseudoconvex and $x \in S$ be a second-order KT point. We prove that x is a global minimizer. Assume that this is not the case and there exists $y \in S$ with $f(y) < f(x)$. We choose the direction d such that $d = y - x$. The direction $y - x$ is critical by the inequalities $\nabla f(x)(y - x) \le 0$ and $\nabla g_i(x)(y - x) \le 0$, $i \in I(x)$. Since x is a KT point there exist $\lambda > 0$ and $\mu_i \ge 0$, $i \in I(x)$ such that (3.1) and (3.2) hold. Then we conclude from $\nabla L(x) = 0$ that

$$\nabla f(x)(y - x) = \nabla g_i(x)(y - x) = 0, \ \forall i \in I(x),$$

such that $\mu_i > 0$. It follows from the pseudoconvexity of (P) that $f''(x; y - x) < 0$ and $g_i''(x; y - x) \le 0$ for all $i \in I(x)$ with $\mu_i > 0$. Hence we obtain that $L''(x; y - x) < 0$ which contradicts (3.2).

Claim (b) follows from Lemma 3.1 and Claim (a). □

Theorem 3.1 (a) is a generalization of Theorems 1 and 3 in Ginchev, Ivanov [14].

Lemma 3.3 *Let the function f be Fréchet differentiable on the open convex set $X \subseteq \mathbf{R}^n$ and second-order directionally differentiable at every point $x \in X$ in every direction $d \in \mathbf{R}^n$.*

(a) *If f is quasiconvex, $x,\ y \in X$, $f(y) < f(x)$, $\nabla f(x) = 0$ and there exists a direction $d \in \mathbf{R}^n$, such that $f''(x; d) < 0$, then $f''(x; y - x) < 0$.*
(b) *If f is strictly quasiconvex, $x,\ y \in X$, $x \ne y$, $f(y) \le f(x)$, $\nabla f(x) = 0$ and there exists a direction $d \in \mathbf{R}^n$, such that $f''(x; d) < 0$, then $f''(x; y - x) < 0$.*

Proof We prove Claim (a). The proof of (b) is similar. Put $z(t) = x + t(y - x)$ with $t \in (0, 1)$. According to the continuity of f there exists $\tau > 0$ and $p = y - \tau d$ such that $f(p) < f(x)$. Let $w(t) = x + \alpha(t)d$ be the point of intersection of the ray $\{x + td \mid t \ge 0\}$ and the straight line passing through p and $z(t)$. An easy calculation gives that $\alpha(t) = t\tau/(1 - t)$. Since f is quasiconvex, we have

$$f(z(t)) \le \max\left(f(p),\ f(w(t))\right) \quad \text{for} \quad 0 < t < 1\,.$$

Therefore

$$t^{-2}(f(z(t)) - f(x)) \le \max\left(t^{-2}(f(p) - f(x)),\ t^{-2}(f(w(t)) - f(x))\right).$$

Since $f(p) < f(x)$, if t tends to 0 with positive values, then the first term of the above maximum tends to $-\infty$. Using that $\nabla f(x) = 0$ we obtain

$$f''(x; y - x) = \lim_{t \to +0} 2t^{-2}(f(z(t)) - f(x)) \le \lim_{t \to +0} 2t^{-2}(f(w(t)) - f(x)).$$

According to the equality

$$\frac{f(w(t)) - f(x)}{t^2} = \frac{f(x + \alpha(t)d) - f(x)}{\alpha^2(t)} \cdot \frac{\alpha^2(t)}{t^2},$$

we obtain that

$$\lim_{t \to +0} 2t^{-2}(f(w(t)) - f(x))$$
$$= \lim_{t \to +0} 2\alpha^{-2}(t)(f(x + \alpha(t)d) - f(x))\,\alpha^2(t)t^{-2} = \tau^2 f''(x; d) < 0.$$

The above inequality yields that $f''(x; y - x) < 0$. □

Theorem 3.2 *Let X be open and convex, f be (strictly) quasiconvex, g_i, $i = 1, 2, \ldots, m$ be quasiconvex. Suppose that f, g_i are Fréchet differentiable and second-order directionally differentiable at every point $x \in S$ in every critical direction $d \in \mathbf{R}^n$. Then the problem* (P) *is second-order (strictly) KT pseudoconvex if and only if each second-order KT point is a (strict) global minimizer.*

Proof We prove the non-strict case. The proof of the strict one is similar.

Let every second-order KT point be a global minimizer. We prove that (P) is second-order KT pseudoconvex. Choose arbitrary $x, y \in S$ with $f(y) < f(x)$. According to the quasiconvexity of f we conclude that $\nabla f(x)(y - x) \leq 0$.

Assume that $\nabla f(x)(y - x) = 0$. By Lemma 2.5 we have $\nabla f(x) = 0$. Suppose that $f''(x; d) \geq 0$ for all critical directions $d \in \mathbf{R}^n$. Then by Lemma 3.2 x is a second-order KT point which implies by the hypothesis that x is a global minimizer. We obtained a contradiction because $y \in S$ and $f(y) < f(x)$. Therefore there exists a critical direction d such that $f''(x; d) < 0$. Then it follows from Lemma 3.3 that $f''(x; y - x) < 0$.

The claim $\nabla g_i(x)(y - x) \leq 0$ and the implication

$$\nabla g_i(x)(y - x) = 0, \ i \in I(x) \quad \text{imply} \quad g_i''(x; y - x) \leq 0$$

follow directly from the assumption $y \in S$, $i \in I(x)$ and the quasiconvexity of g_i.

The only if part is a Theorem 3.1 (a). □

Theorem 3.3 *Let the set X be open and convex. Suppose that the functions f, g_i, $i = 1, 2, \ldots, m$ are defined on X, differentiable and second-order directionally differentiable at every feasible point x in every critical direction $d \in \mathbf{R}^n$. If f is (strictly) quasiconvex and each KT point of second-order is a (strict) global minimizer, then f is second-order (strictly) pseudoconvex on S.*

Proof We can prove the theorem using the arguments of Theorem 3.2. □

3.2 Second-order FJ pseudoconvex problems

We introduce the following definition:

Definition 3.6 We call the problem (P) second-order (strictly) FJ pseudoconvex at the point $x \in S$ if the following implication holds

$$\left.\begin{matrix} y \in S, \\ f(y) < f(x) \end{matrix}\right] \Rightarrow \left[\begin{matrix} \nabla f(x)(y-x) \leq 0, \\ \nabla f(x)(y-x) = 0 \;\text{ implies }\; f''(x; y-x) < 0, \\ \nabla g_i(x)(y-x) \leq 0 \;\text{ for all }\; i \in I(x), \\ \nabla g_i(x)(y-x) = 0,\; i \in I(x) \;\text{ imply }\; g_i''(x; y-x) < 0. \end{matrix}\right.$$

$$\left(\left.\begin{matrix} y \in S,\; y \neq x, \\ f(y) \leq f(x) \end{matrix}\right] \Rightarrow \left[\begin{matrix} \nabla f(x)(y-x) \leq 0, \\ \nabla f(x)(y-x) = 0 \;\text{ implies }\; f''(x; y-x) < 0, \\ \nabla g_i(x)(y-x) \leq 0 \;\text{ for all }\; i \in I(x), \\ \nabla g_i(x)(y-x) = 0,\; i \in I(x) \;\text{ imply }\; g_i''(x; y-x) < 0. \end{matrix}\right. \right)$$

provided that all necessary derivatives exist. We call (P) second-order (strictly) FJ pseudoconvex if it is second-order (strictly) FJ pseudoconvex at each $x \in S$.

It is obvious that every problem which is (strictly) FJ pseudoconvex with respect to the Fréchet derivative, is second-order (strictly) FJ pseudoconvex. Really, the existence of the second-order directional derivative in this case is not required. Each second-order (strictly) FJ pseudoconvex problem is second-order (strictly) KT pseudoconvex.

Like FJ pseudoconvex problems, second-order FJ pseudoconvex ones possess some useful properties.

Theorem 3.4 *Let X be an open convex set. Suppose that f, g_i, $i = 1, 2, \ldots, m$ are Fréchet differentiable and second-order directionally differentiable at any point $x \in S$ in every critical direction d. If* (P) *is second-order strictly FJ pseudoconvex, then the feasible set S is convex.*

Proof The proof is similar to the proof of Theorem 2.5. We suppose that there exist $x, y \in S$ and $z \in (x, y)$, $z \notin S$. Using the arguments of Theorem 2.5 we obtain that there exist $u, v \in S$, an index $j \in \{1, 2, \ldots, m\}$ and a sequence $\{u_k\}$, $u_k \in (u, v)$, $u_k \to u$ such that $z \in (u, v)$, $(u, v) \notin S$, $g_j(u) = 0$ and $g_j(u_k) > 0$. Without loss of generality $f(v) \leq f(u)$. It follows from the second-order strict FJ pseudoconvexity that $\nabla g_j(u)(v-u) < 0$ or $\nabla g_j(u)(v-u) = 0$, $g_j''(u; v-u) < 0$. In both cases there exists $\delta > 0$ with $g_j(u + t(v-u)) < 0$ for all $t \in (0, \delta)$ which is a contradiction. □

Theorem 3.5 *Let X be an open set. Suppose that f, g_i, $i = 1, 2, \ldots, m$ are Fréchet differentiable and second-order directionally differentiable at any point $x \in S$ in every critical direction d.*

(a) *If* (P) *is second-order (strictly) FJ pseudoconvex, then every FJ point of second-order is a (strict) global minimizer.*
(b) *If* (P) *is second-order FJ pseudoconvex, then every local minimizer of* (P) *is global, provided that $f, g \in C^1(X)$.*
(c) *If* (P) *is second-order strictly FJ pseudoconvex, then every local minimizer of* (P) *is global.*

Proof The proof of Claim (a) follows the arguments of Theorem 3.1.

Claim (b) follows from Lemma 3.1 and Claim (a).

Claim (c). We conclude from second-order strict FJ pseudoconvexity that f is second-order strictly pseudoconvex. It follows from Theorem 3.4 that the feasible set S is convex. Then by Proposition 3.2 f is strictly quasiconvex. Therefore by the definition of strictly quasiconvex functions every local minimizer of (P) is a global minimizer. □

Theorem 3.6 *Let the set X be open and convex. Suppose that the functions f, g_i, $i = 1, 2, \ldots, m$ are defined on X, differentiable and second-order directionally differentiable at every feasible point x in every critical direction $d \in \mathbf{R}^n$. Suppose that f is (strictly) quasiconvex on X, g_i, $i = 1, 2, \ldots, m$ are strictly quasiconvex on X. Then* (P) *is second-order (strictly) FJ pseudoconvex if and only if each second-order FJ point is a (strict) global minimizer.*

Proof We prove the non-strict case. Suppose that every second-order FJ point is a global minimizer. We prove that (P) is second-order pseudoconvex. Let $x, y \in S$ and $f(y) < f(x)$.

It follows from the quasiconvexity of f and $f(y) < f(x)$ that $\nabla f(x)(y - x) \leq 0$.

Assume that $\nabla f(x)(y - x) = 0$. Applying the arguments of Theorem 3.2 it follows from Lemmas 3.2 and 3.3 that $\nabla f(x) = 0$ and $f''(x; y - x) < 0$.

The inequality $\nabla g_i(x)(y - x) \leq 0$ follows from the quasiconvexity of g_i and $y \in S$, $i \in I(x)$ because each strictly quasiconvex function is quasiconvex.

Let $\nabla g_i(x)(y - x) = 0, i \in I(x)$. By the strict quasiconvexity of g_i we have $g_i(z) < g_i(x)$ for all $z \in (x, y)$. Then we prove that $g_i''(x; y - x) < 0$ applying Lemmas 3.2, 3.3 and the other arguments of Theorem 3.2, replacing y by z and f by g_i.

The only if part of the proof follows from Theorem 3.5 (a). □

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